Int. J. Solids Structures, 1978, Vol. 14, pp. 131–139. Pergamon Press. Printed in Great Britain

BOND STRESSES FOR A COMPOSITE MATERIAL REINFORCED WITH VARIOUS ARRAYS OF FIBERS

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(*Received* 30 *August* 1976; *revised* 25 *July* 1977)

Abstract—The problem treated here is that of an isotropic body having a doubly periodic rectangular or triangular array of perfectly bonded circular elastic inclusions. The body is in tension or compression. This simulates a composite material wherein a relatively weak matrix is reinforced by stronger (and more rigid) fibers. Bond stresses for both rectangular and triangular arrays have been calculated using either boundary point matching or boundary point least squares techniques. Numerical results based on a plane strain analysis are given in graphical form.

INTRODUCTION

The potential use of fiber reinforced composites in developing light weight, high strength engineering materials is well recognized. A great deal of current research, both analytical and experimental, has been aimed at a better understanding of the mechanical behavior of these materials.

An important consideration in the analysis of fiber reinforced materials is the nature and magnitude of the bond stresses between the relatively rigid fibers and the flexible matrix. A knowledge of these quantities is required for design purposes. Thus it is the purpose of the present investigation to obtain results for one such problem, simulated as a body of isotropic matrix material reinforced by a doubly periodic array of perfectly bonded circular elastic inclusions (the fibers). In order to have as much generality as possible, the fibers are assumed to be based either on a rectangular or an isosceles traingular array. A state of plane strain is assumed throughout.

A few other investigations have been made on related problems. For example, Fil'shtinskii[1] and Grigolyuk and Fil'shtinskii[2] treated similar problems to the ones discussed here using a very complicated method which is most difficult to follow. They were mostly concerned with obtaining effective Young's moduli although they did give a few results on bond stresses.

Using point matching and point least squares methods, Wilson and Hill[3] have presented an analysis of an infinite plate with a doubly periodic array of circular or elliptical holes or rigid inclusions. Their results are only for rectangular arrays, which is the simplest configuration to analyze.

The present more general analysis is based upon a complex variable formulation[4] in conjunction with the point matching method[5–7] or the point least squares method[8]. In the problems investigated, perfect bonding between fibers and matrix is assumed, the continuity stress and displacement conditions occurring at the interfaces.

It should be mentioned that, in dealing with problems of the transverse properties of reinforced concrete, the present analysis can also be applied. The matrix in reinforced concrete is the concrete, and the fibers are the steel rods.

COMPLEX VARIABLE FORMULATION

When the complex variable method is applied to the plane theory of elasticity, the problem of solving for the Airy stress function is one of determining two analytic functions, $\Phi(z)$ and $\Psi(z)$[4]. The displacements and stresses can then be expressed in the respective forms

$$2\mu(u+iv) = h\phi(z) - z\overline{\Phi(z)} - \overline{\psi(z)} = 2\mu(v_r + iv_\theta)\,e^{i\theta} \quad (1)$$

$$\sigma_x + \sigma_y = 2[\Phi(z) + \overline{\Phi(z)}] = \sigma_r + \sigma_\theta \quad (2)$$

$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2[\bar{z}\Phi'(z) + \Psi(z)] = (\sigma_\theta - \sigma_r + 2i\tau_{r\theta})^{-i2\theta} \quad (3)$$

and

$$\sigma_r - i\tau_{r\theta} = \Phi(z) + \overline{\Phi(z)} - e^{i2\theta}[\bar{z}\Phi'(z) + \Psi(z)] \quad (4)$$

where

$$\phi(z) = \int \Phi(z)\,dz \text{ and } \psi(z) = \int \Psi(z)\,dz. \tag{5}$$

Here u, v, σ_x, σ_y and τ_{xy} are displacement components and stress components respectively in Cartesian coordinates, while v_r, v_θ, σ_r, σ_θ and $\tau_{r\theta}$ are those in a polar coordinate system. Also μ is the shear modulus and h is equal to $(3–4\nu)$ for plane strain, ν being Poisson's ratio.

The doubly periodic arrays treated here can be classified into two typical configurations, as shown in Fig. 1, namely rectangular and triangular. The ratio between the moduli of elasticity of the inclusion and the matrix is assumed to have any value. If the ratio is zero, we have the case of a plate weakened by holes. If the ratio approaches infinity, then the plate is reinforced by rigid inclusions.

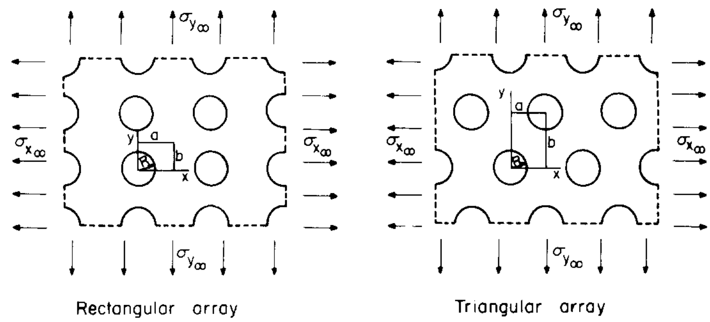

Fig. 1. Plate with doubly periodic arrays of circular inclusions.

In this investigation, quantities with subscript 1 refer to the inclusion, and those with subscript 2 refer to the matrix. By assuming that the plate is infinite, and the stresses at infinity are applied along the coordinate axes, it follows by symmetry that $u(z) = u(\bar{z}) = -u(-\bar{z})$, $v(z) = -v(\bar{z}) = v(-\bar{z})$, $u = 0$ and $\tau_{xy} = 0$ on $x = 0$, and $v = 0$ and $\tau_{xy} = 0$ on $y = 0$. We can then analyze the problem by considering only a quarter of a unit element, as shown in Fig. 1, with appropriate complex functions for inclusion and matrix. We then write

$$\left.\begin{aligned} \Phi_1(z) &= \sum_{k=1}^{l} a_k z^{2k-2}, \quad \Psi_1(z) = \sum_{k=1}^{m} a'_k z^{2k-2} \\ \Phi_2(z) &= \sum_{k=-n}^{r} b_k z^{2k}, \quad \Psi_2(z) = \sum_{k=-s}^{t} b'_k z^{2k} \end{aligned}\right\} \tag{6}$$

where a_k, a'_k, b_k and b'_k are real constants and l, m, n, r, s, and t are arbitrary integers. It is noted that Φ_1 and Ψ_1 are analytic in the simply connected region $|z| \le R$, while Φ_2 and Ψ_2 are analytic in the doubly connected region $|z| \ge R$, $|x| \le a$, $|y| \le b$.

Because the inclusions are assumed to be perfectly bonded to the matrix, the stress continuity condition across $|z| = R$ requires that

$$(\sigma_r - i\tau_{r\theta})_1 = (\sigma_r - i\tau_{r\theta})_2. \tag{7}$$

Substitution of eqn (4) into eqn (7) yields

$$(\Phi(z) + \overline{\Phi(z)} - e^{i2\theta}[\bar{z}\Phi'(z) + \Psi(z)])_1 = (\Phi(z) + \overline{\Phi(z)} - e^{i2\theta}[\bar{z}\Phi'(z) + \Psi(z)])_2 \tag{8}$$

and with $z = Re^{i\theta}$ substituted into eqn (8) and by comparing the coefficients of the terms $e^{i2k\theta}$, we obtain

$$\left.\begin{aligned} 2a_1 &= 2b_0 - R^{-2}b'_{-1} \\ (1-2k)a_{k+1} - R^{-2}a'_k &= (1-2k)b_k + R^{-4k}b_{-k} - R^{-2}b'_{k-1},\ k \ge 1 \\ R^{4k}a_{k+1} &= (1+2k)b_{-k} + R^{4k}b_k - R^{-2}b'_{-(k+1)},\ k \ge 1 \end{aligned}\right\}. \tag{9}$$

Also, on $|z| = R$, the displacement continuity condition requires

$$(v_r + iv_\theta)_1 = (v_r + iv_\theta)_2 \text{ or } (u + iv)_1 = (u + iv)_2 \tag{10}$$

it follows that

$$\left.\begin{aligned}
(h_1 - 1)\mu_2 R a_1 &= (h_2 - 1)\mu_1 R b_0 + \mu_1 R^{-1} b'_{-1} \\
R^{4k} a_{k+1} &= \frac{\mu_1 h_2}{\mu_2 h_1} R^{4k} b_k - (2k+1)\frac{\mu_1}{\mu_2 h_1} b_{-k} + \frac{\mu_1}{\mu_2 h_1} R^{-2} b'_{-(k+1)},\ k \geq 1 \\
(1-2k)a_{k+1} - R^{-2} a'_k &= -\frac{\mu_1 h_2}{\mu_2} R^{-4k} b_{-k} + \frac{\mu_1}{\mu_2}(1-2k) b_k - \frac{\mu_1}{\mu_2} R^{-2} b'_{k-1},\ k \geq 1
\end{aligned}\right\}. \tag{11}$$

From eqns (9) and (11), we express b'_k in terms of b_k in the forms

$$\left.\begin{aligned}
b'_{-1} &= S_1 b_0 \\
b'_{-(k+1)} &= R^2(1+2k) b_{-k} + S_2 R^{4k+2} b_k,\ k \geq 1 \\
b'_{k-1} &= R^2(1-2k) b_k + S_3 R^{2-4k} b_{-k},\ k \geq 1
\end{aligned}\right\} \tag{12}$$

where

$$\left.\begin{aligned}
S_1 &= \frac{2R^2[(h_1-1)\mu_{21} - (h_2-1)]}{2 + (h_1-1)\mu_{21}} \\
S_2 &= \frac{\mu_{21} h_1 - h_2}{\mu_{21} h_1 + 1} \\
S_3 &= \frac{\mu_{21} + h_2}{\mu_{21} - 1},\ \mu_{21} = \frac{\mu_2}{\mu_1}
\end{aligned}\right\}. \tag{13}$$

The problem can then be solved if the values of b_k are determined.

To compute for this set of unknowns b_k, $k = -q, p$, where q and p are arbitrarily chosen depending on the accuracy required, the boundary conditions must be applied. Then either the point matching or the point least squares method can be used to solve for the values of b_k.

From eqns (1)–(3) and the approximate forms for Φ_2 and Ψ_2, we express $\sigma_x)_2$, $\sigma_y)_2$, $\tau_{xy})_2$, u_2 and v_2 in terms of b_k in the forms

$$\left.\begin{aligned}
\tau_{xy})_2 &= Im[\bar{z}\Phi'_2 + \Psi_2] = \sum_{k=-q}^{p} D_k b_k \\
\sigma_x)_2 &= Re[\Phi_2 + \bar{\Phi}_2 - \bar{z}\Phi'_2 - \Psi_2] = \sum_{k=-q}^{p} E_k b_k \\
\sigma_y)_2 &= Re[\Phi_2 + \bar{\Phi}_2 + \bar{z}\Phi'_2 + \Psi_2] = \sum_{k=-q}^{p} F_k b_k \\
2\mu u_2 &= Re[h_2\phi_2 - z\bar{\Phi}_2 - \psi_2] = \sum_{k=-q}^{p} G_k b_k \\
2\mu v_2 &= Im[h_2\phi_2 - z\bar{\Phi}_2 - \psi_2] = \sum_{k=-q}^{p} H_k b_k
\end{aligned}\right\} \tag{14}$$

where

$$\left.\begin{aligned}
D_k &= Im\{S_1 z^{-2}\},\ k=0 & G_k &= Re\{h_2 z - z + S_1\bar{z}^{-1}\},\ k=0 \\
& Im\{DC1_k\},\ -q \leq k \leq -1 & & Re\{GC1_k\},\ -q \leq k \leq -1 \\
& Im\{DC2_k\},\ 1 \leq k \leq p & & Re\{GC2_k\},\ 1 \leq k \leq p \\
E_k &= Re\{2 - S_1 z^{-2}\},\ k=0 & H_k &= Im\{h_2 z - z + S_1\bar{z}^{-1}\},\ k=0 \\
& Re\{2z^{2k} - DC1_k\},\ -q \leq k \leq -1 & & Im\{GC1_k\},\ -q \leq k \leq -1 \\
& Re\{2z^{2k} - DC2_k\},\ 1 \leq k \leq p & & Im\{GC2_k\},\ 1 \leq k \leq p \\
F_k &= Re\{2 + S_1 z^{-2}\},\ k=0 & & \\
& Re\{2z^{2k} + DC1_k\},\ -q \leq k \leq -1 & & \\
& Re\{2z^{2k} + DC2_k\},\ 1 \leq k \leq p & &
\end{aligned}\right\} \tag{15}$$

and where

$$\left.\begin{aligned} DC1_k &= 2k\bar{z}z^{2k-1} + S_3 R^{2(2k+1)} z^{-(2k+2)} - R^2(2k-1)z^{2k-2} \\ DC2_k &= 2k\bar{z}z^{2k-1} + S_2 R^{2(2k+1)} z^{-(2k+2)} - R^2(2k-1)z^{2k-2} \\ GC1_k &= \frac{h_2}{2k+1} z^{2k+1} - z\bar{z}^{2k} + R^2 \bar{z}^{2k-1} + \frac{S_3}{2k+1} R^{2(2k+1)} \bar{z}^{-(2k+1)} \\ GC2_k &= \frac{h_2}{2k+1} z^{2k+1} - z\bar{z}^{2k} + R^2 \bar{z}^{2k-1} + \frac{S_2}{2k+1} R^{2(2k+1)} \bar{z}^{-(2k+1)} \end{aligned}\right\}. \quad (16)$$

The boundary conditions are now applied using either the point matching or the point least squares method to the rectangular and triangular arrays, separately.

1. *Rectangular array*

Consider the first quadrant of a unit element, as illustrated in Fig. 2(a).

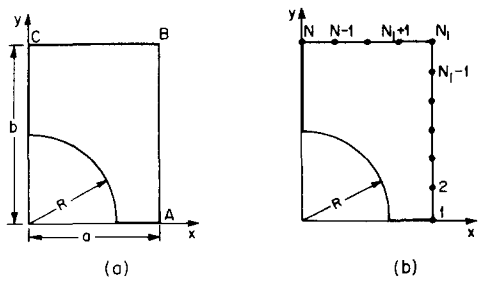

Fig. 2. Typical element in rectangular array.

Boundary conditions.

$$\left.\begin{aligned} \tau_{xy} &= 0 \quad \text{on } AB \text{ and } BC \\ u &= d_1 \quad \text{on } AB \\ v &= d_2 \quad \text{on } BC \end{aligned}\right\} \quad (17)$$

where d_1 and d_2 are the constants to be determined such that the average stress on AB is $\sigma_{x\infty}$ and that on BC is $\sigma_{y\infty}$.

Point matching method. This method requires that the boundary conditions be satisfied exactly at the n boundary points shown in Fig. 2(b). It follows from eqns (17) that

$$\left.\begin{aligned} &\sum_{k=-q}^{p} D_k(z_i) b_k = 0, && 2 \le i \le N-1 \\ &\sum_{k=-q}^{p} G_k(z_i) b_k = d_1, && 1 \le i \le N_1 \\ \text{and} \quad &\sum_{k=-q}^{p} H_k(z_i) b_k = d_2, && N_1 \le i \le N \end{aligned}\right\}. \quad (18)$$

It is clear that the number of unknown coefficients $p+q+1$ must be equal to the number of algebraic equation $2n-1$.

The procedures to solve for b_k, $k = -q, p$, from Ref. [3], can be described as follows. Let b_{k_1} denote the solution for $d_1 = 1$ and $d_2 = 0$, and those for $d_1 = 0$ and $d_2 = 1$ be denoted b_{k_2}. Also, let $\sigma_{x\infty 1}$ and $\sigma_{y\infty 1}$ denote the average normal stresses on AB and BC respectively with respect to the solution b_{k_1}, and those with respect to b_{k_2} be $\sigma_{x\infty 2}$ and $\sigma_{y\infty 2}$. Then, by superposition, we obtain the solution b_k for any specified values of $\sigma_{x\infty}$ and $\sigma_{y\infty}$.

$$b_k = \left(\frac{\sigma_{x\infty}\sigma_{y\infty 2} - \sigma_{y\infty}\sigma_{x\infty 2}}{\sigma_{x\infty 1}\sigma_{y\infty 2} - \sigma_{x\infty 2}\sigma_{y\infty 1}}\right) b_{k_1} + \left(\frac{\sigma_{y\infty}\sigma_{x\infty 1} - \sigma_{x\infty}\sigma_{y\infty 1}}{\sigma_{x\infty 1}\sigma_{y\infty 2} - \sigma_{x\infty 2}\sigma_{y\infty 1}}\right) b_{k_2}. \quad (19)$$

Point least squares method. Unlike the point matching method, this method does not require that the boundary conditions be exactly satisfied at the n selected points. Instead, it minimizes the squares of errors in satisfying the boundary conditions at those points. That is, we minimize the function

$$U = \lambda_1 \sum_{i=2}^{N-1} \left[\sum_{k=-q}^{p} D_k(z_i) b_k \right]^2 + \lambda_2 \sum_{i=1}^{N_1} \left[\sum_{k=-q}^{p} G_k(z_i) b_k - d_1 \right]^2 + \lambda_2 \sum_{i=N_1}^{N} \left[\sum_{k=-q}^{p} H_k(z_i) b_k - d_2 \right]^2. \quad (20)$$

Usually, as is done here, the values of λ are set equal to unity. Then, during minimization of U, we obtain a set of algebraic equations

$$\sum_{k=-q}^{p} \left[\sum_{i=2}^{N-1} D_k(z_i) D_j(z_i) + \sum_{i=1}^{N_1} G_k(z_i) G_j(z_i) + \sum_{i=N_1}^{N} H_k(z_i) H_j(z_i) \right] b_k$$
$$= \sum_{i=1}^{N_1} d_1 G_j(z_i) + \sum_{i=N_1}^{N} d_2 H_j(z_i), \quad -q \le j \le p \quad (21)$$

from which b_k, $k = -q, p$, can be solved by following the same superposition procedures as previously described. In this method, $(p + q + 1)$ does not have to be equal to $(2n - 1)$.

2. *Triangular array*

To avoid introducing functions other than Φ_2 and Ψ_2 in the computation, we consider the problem with a repeating element $ABCDEF$, as shown in Fig. 3(a).

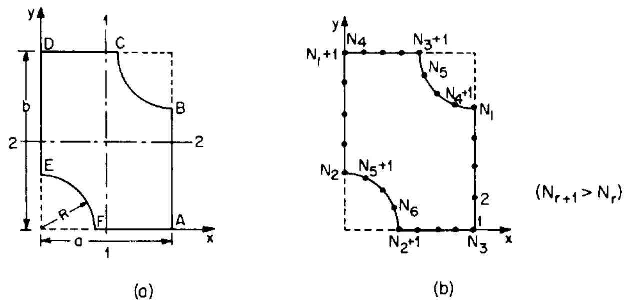

Fig. 3. Typical element in triangular array.

Boundary conditions. By the symmetry of the problem, it is required to apply the following necessary and sufficient conditions

$$\left.\begin{array}{l}
\tau_{xy}, \sigma_x \text{ and } \sigma_y \text{ on } AB = \tau_{xy}, \sigma_x \text{ and } \sigma_y \text{ on } DE, \text{ respectively, at the corresponding points} \\
\tau_{xy}, \sigma_x \text{ and } \sigma_y \text{ on } FA = \tau_{xy}, \sigma_x \text{ and } \sigma_y \text{ on } CD, \text{ respectively, at the corresponding points} \\
\tau_{xy}, \sigma_x \text{ and } \sigma_y \text{ on } BC = \tau_{xy}, \sigma_x \text{ and } \sigma_y \text{ on } EF, \text{ respectively, at the corresponding points} \\
u = d_1 \text{ on } AB \\
v = d_2 \text{ on } CD
\end{array}\right\} \quad (22)$$

where d_1 and d_2 are determined such that the average normal stress on line $1-1$ is $\sigma_{x\infty}$, and that on line $2-2$ is $\sigma_{y\infty}$.

Point matching method. With the configuration shown in Fig. 3(b), and corresponding to the boundary conditions given above, the point matching method yields the equations

$$\left.\begin{aligned}
&\sum_{k=-q}^{p} D_k(z_i)b_k = \sum_{k=-q}^{p} D_k(z_{i+N_1})b_k, \qquad 2 \le i \le N_1\\
&\sum_{k=-q}^{p} E_k(z_i)b_k = \sum_{k=-q}^{p} E_k(z_{i+N_1})b_k, \qquad 1 \le i \le N_1\\
&\sum_{k=-q}^{p} F_k(z_i)b_k = \sum_{k=-q}^{p} F_k(z_{i+N_1})b_k, \qquad 1 \le i \le N_1\\
&\sum_{k=-q}^{p} D_k(z_i)b_k = \sum_{k=-q}^{p} D_k(z_{i+N_{23}})b_k, \qquad N_2+1 \le i \le N_3-1\\
&\sum_{k=-q}^{p} E_k(z_i)b_k = \sum_{k=-q}^{p} E_k(z_{i+N_{23}})b_k, \qquad N_2+1 \le i \le N_3-1\\
&\sum_{k=-q}^{p} F_k(z_i)b_k = \sum_{k=-q}^{p} F_k(z_{i+N_{23}})b_k, \qquad N_2+1 \le i \le N_3-1\\
&\sum_{k=-q}^{p} D_k(z_i)b_k = \sum_{k=-q}^{p} D_k(z_{i+N_{45}})b_k, \qquad N_4+1 \le i \le N_5\\
&\sum_{k=-q}^{p} E_k(z_i)b_k = \sum_{k=-q}^{p} E_k(z_{i+N_{45}})b_k, \qquad N_4+1 \le i \le N_5\\
&\sum_{k=-q}^{p} F_k(z_i)b_k = \sum_{k=-q}^{p} F_k(z_{i+N_{45}})b_k, \qquad N_4+1 \le i \le N_5\\
&\sum_{k=-q}^{p} G_k(z_i)b_k = d_1, \qquad 1 \le i \le N_1\\
&\sum_{k=-q}^{p} H_k(z_i)b_k = d_2, \qquad N_3+1 \le i \le N_4
\end{aligned}\right\} \tag{23}$$

where $N_{23} = N_3 - N_2$ and $N_{45} = N_5 - N_4$. This yields $4N_1 + 4N_{23} + 3N_{45} - 4$ equations. By using the same procedure as that described in problem 1, except that the average normal stresses are now taken along the lines 1–1 and 2–2, we can then obtain the solution b_k from eqn (19).

Point least squares method. With all the λ's placed equal to unity, and after minimization of the squares of the errors, we obtain

$$\left.\begin{aligned}
&\sum_{k=-q}^{p}\Bigg\{\sum_{i=2}^{N_1}(D_k(z_i)[D_j(z_i)-D_j(z_{i+N_1})]+D_k(z_{i+N_1})[D_j(z_{i+N_1})-D_j(z_i)])\\
&\quad+\sum_{i=1}^{N_1}(E_k(z_i)[E_j(z_i)-E_j(z_{i+N_1})]+E_k(z_{i+N_1})[E_j(z_{i+N_1})-E_j(z_i)])\\
&\quad+\sum_{i=1}^{N_1}(F_k(z_i)[F_j(z_i)-F_j(z_{i+N_1})]+F_k(z_{i+N_1})[F_j(z_{i+N_1})-F_j(z_i)])\\
&\quad+\sum_{i=N_2+1}^{N_3-1}(D_k(z_i)[D_j(z_i)-D_j(z_{i+N_{23}})]+D_k(z_{i+N_{23}})[D_j(z_{i+N_{23}})-D_j(z_i)])\\
&\quad+\sum_{i=N_2+1}^{N_3-1}(E_k(z_i)[E_j(z_i)-E_j(z_{i+N_{23}})]+E_k(z_{i+N_{23}})[E_j(z_{i+N_{23}})-E_j(z_i)])\\
&\quad+\sum_{i=N_2+1}^{N_3-1}(F_k(z_i)[F_j(z_i)-F_j(z_{i+N_{23}})]+F_k(z_{i+N_{23}})[F_j(z_{i+N_{23}})-F_j(z_i)])\\
&\quad+\sum_{i=N_4+1}^{N_5}(D_k(z_i)[D_j(z_i)-D_j(z_{i+N_{45}})]+D_k(z_{i+N_{45}})[D_j(z_{i+N_{45}})-D_j(z_i)])\\
&\quad+\sum_{i=N_4+1}^{N_5}(E_k(z_i)[E_j(z_i)-E_j(z_{i+N_{45}})]+E_k(z_{i+N_{45}})[E_j(z_{i+N_{45}})-E_j(z_i)])\\
&\quad+\sum_{i=N_4+1}^{N_5}(F_k(z_i)[F_j(z_i)-G_j(z_{i+N_{45}})]+F_k(z_{i+N_{45}})[F_j(z_{i+N_{45}})-F_j(z_i)])\\
&\quad+\sum_{i=1}^{N_1}(G_k(z_i)G_j(z_i))+\sum_{i=N_3+1}^{N_4}(H_k(z_i)H_j(z_i))\Bigg\}b_k=\sum_{i=1}^{N_1}d_1G_j(z_i)+\sum_{i=N_3+1}^{N_4}d_2H_j(z_i), \quad -q\le j\le p
\end{aligned}\right\} \tag{24}$$

NUMERICAL RESULTS

Some numerical results are presented for both rectangular and triangular arrays in graphical form. The results are obtained for the case of plane strain, and the Poisson's ratios for both matrix and inclusion (fiber) are assumed to be 0.3. Also, the stresses at infinity, $\sigma_{x\infty}$ and $\sigma_{y\infty}$, are set equal to 1 and 0 respectively.

To examine the influence on the bond stresses due to the spacing of inclusions, the results are illustrated in Figs. 4–7 for two square arrays with different values of a/R. In Figs. 8 and 9, bond stresses are evaluated for a regular triangular array (b/a = 1.732).

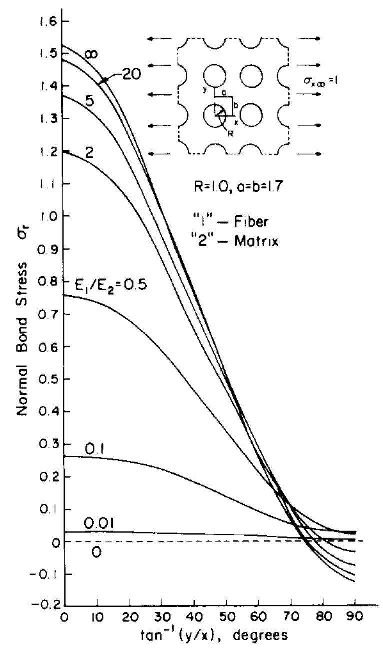

Fig. 4. Normal bond stress on a square array (R = 1, a = b = 1.7).

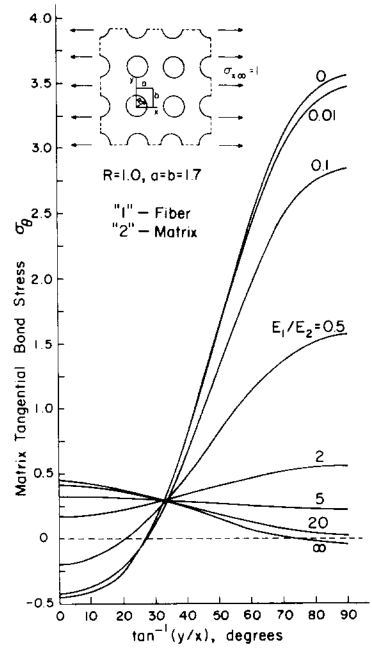

Fig. 5. Matrix tangential bond stress on a square array (R = 1, a = b = 1.7).

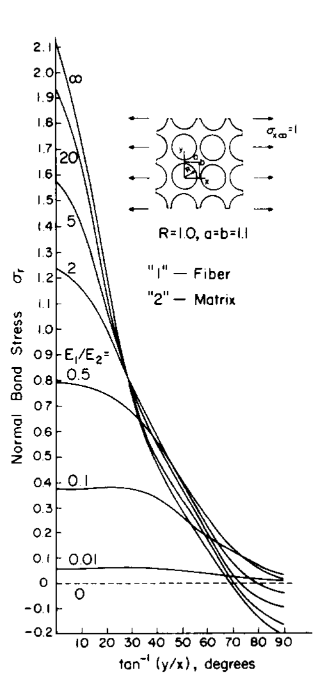

Fig. 6. Normal bond stress on a square array (R = 1, a = b = 1.1).

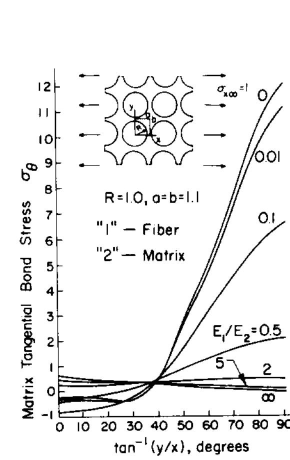

Fig. 7. Matrix tangential bond stress on a square array (R = 1, a = b = 1.1).

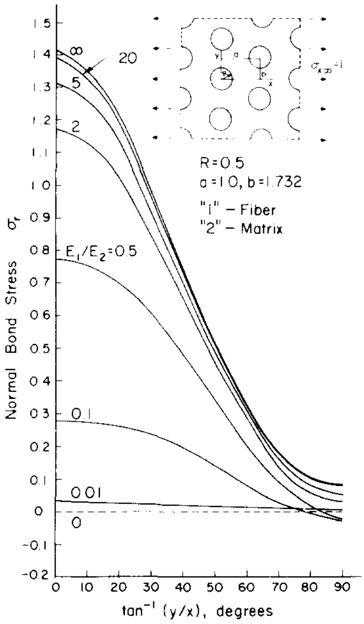

Fig. 8. Normal bond stress on a triangular array ($R = 0.5$, $a = 1$, $b = 1.732$).

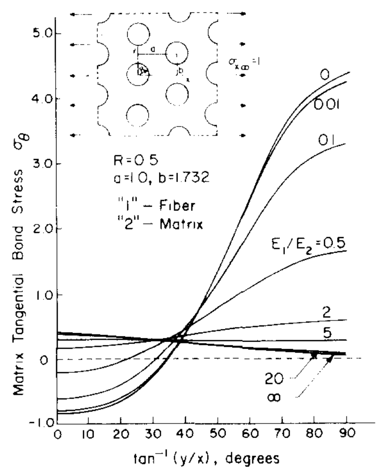

Fig. 9. Matrix tangential bond stress on a triangular array ($R = 0.5$, $a = 1$, $b = 1.732$).

CONCLUSIONS

Both the point matching method and the point least squares method gave very good results. Since the former method usually requires more terms in Φ and Ψ than the latter does, thus, when the number of boundary points becomes large, it is better to use the point least squares method with less unknown coefficients.

Also, to prevent overflow or underflow in solving for a large system of linear algebraic equations, it is possible to adjust the integers p and q, as a result of experience, and then divide both right hand side and left hand side of the equations by an optimum number, which may be chosen as the square root of the first term in one of the equations.

Some of the cases presented in this paper have also been solved by the finite element method and the corresponding results agree very well. However, the computing expense is far lower in the present analysis. The finite element method provides a lower bound solution, that is, under a given load, the approximate plate behaves stiffer than the actual plate does, and the approximate displacement solution bounds the exact solution from below.

For the case of a plate weakened by holes, when the values of both a/R and b/R are greater than 3, the present analysis shows that the problem can be treated as one of an infinite plate weakened by a single hole. That is, the holes do not interfere stresswise with one another. This analysis also indicates that when the value of E_1/E_2 is less than 0.05, the existence of the presence of the fibers can be ignored. Finally, when the value of E_1/E_2 is greater than 20, the fibers can be considered rigid.

For arbitrarily shaped inclusions, the present analysis must be accompanied by conformal mapping techniques. With appropriate mapping functions, which transform the inside and outside of a circle to the inside and outside of the required shape, we can then obtain the solution for a plate with a doubly periodic array of arbitrarily shaped inclusions.

Acknowledgement—The financial support of the National Science Foundation through the Materials Science Center, Cornell University is gratefully acknowledged.

REFERENCES

1. L. A. Fil'shtinskii, Stresses and displacements in an elastic plane weakened by a doubly periodic system of identical circular holes. *Appl. Math. and Mech.* (Moscow), **28**, 430–441 (1964).
2. E. I. Grigolyuk and L. A. Fil'shtinskii, Elastic equilibrium of an isotropic plane with a doubly periodic system of inclusions. *Prikladnaya Mekhanika* **2**(9), 1–7 (1966).
3. H. B. Wilson, Jr. and J. L. Hill, *Plane Elastostatic Analysis of an Infinite Plate with a Doubly Periodic Array of Holes or Rigid Inclusions.* Rohm & Haas Company, Report No. S-50 (June 1965).

4. N. I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity*. Noordhoff, Groningen, Holland (1963).
5. H. D. Conway, The approximate analysis of certain boundary value problems. *J. Appl. Mech.* **27**, Trans. ASME. **82**, Series E, 275–277 (June 1960).
6. M. D. Heaton, A calculation of the elastic constants of a unidirectional fibre-reinforced composite. *J. Physics*. Ser. 2, **1D**, 1039–1048 (1968).
7. A. W. Leissa, W. E. Clausen and G. K. Agrawal, Stress and deformation analysis of fibrous composite materials by point matching, *Int. J. Num. Methods Engng* **3**, 89–101 (1971).
8. W. L. Chu and H. D. Conway, A numerical method for computing the stresses around an axisymmetrical inclusion. *Int. J. Mech. Sci.* **12**, 575–588 (1970).